关于 k阶 Smarandache ceil函数与 $a_k(n)$的渐进公式*

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摘　要:利用解析方法来研究 k阶 Smarandache ceil函数作用在 k次方根 $a_k(n)$上的均值,从而得出几个有趣的渐进公式.

关键词: Smarandache ceil函数; k次方根 $a_k(n)$;渐进公式

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1　引言及结论

设 n为正整数,$a_k(n)$表示 n的 k次方根的整数部分,即:

$a_k(n) = [n^{\frac{1}{k}}]$　$n = 0, 1, 2, 3\cdots$,其中 $[n]$为不超过 n的最大整数.

著名的 k阶 Smarandache ceil函数定义如下:

$S_k(n) = \min\{x \in N \mid n \mid x^k\}$　$(\forall n \in N^*)$

这两个函数都是 F. Smarandache教授在文献[1]中提出的,许多学者对其产生浓厚的兴趣,但至今,k阶 Smarandache ceil函数与 k次方根 $a_k(n)$之间的关系似乎尚未有人研究.本文主要研究 k阶 Smarandache ceil函数作用在 k次方根 $a_k(n)$上的均值,从而得出下面几个有趣的渐进公式.

定理 1　对任意实数 $x \geq 1, k \geq 3$,我们有

$$\sum_{n \leq x} S_k(a_k(n)) = \frac{1}{k} Y(k-1) g(1) x + O(x^{1-\frac{1}{2k}+X})$$

定理 2　对任意实数 $x \geq 1, k \leq 2$,我们有

$$\sum_{n \leq x} S_k(a_k(n)) = \frac{k}{k^2-k+2} Y(\frac{2}{k}) g(\frac{2}{k}) x^{\frac{k^2-k+2}{k^2}} + O(x^{\frac{k^2-k+2}{k^2}+X})$$

其中 $Y(s)$为 Riemann Zeta函数,X为任意正数,

$$g(s) = \prod_p (1 + p^{1-s} - p^{1-ks} - p^{-s})$$

2　引理及证明

引理 1　对任意实数 $x \geq 1$,我们有

$$\sum_{n \leq x} S_k(n) = \begin{cases} Y(k-1) g(1) x + O(x^{\frac{1}{2}+X}) & k \geq 3 \\ \frac{k}{2} Y(\frac{2}{k}) g(\frac{2}{k}) x^{\frac{2}{k}} + O(x^{\frac{2}{k}+X}) & k \leq 2 \end{cases}$$

证明:令

$$f(s) = \sum_{n=1}^{\infty} \frac{S_k(n)}{n^s} =$$

$$\prod_p \left(1 + \frac{S_k(p)}{p^s} + \cdots + \frac{S_k(p^k)}{p^{ks}} + \frac{S_k(p^{(k+1)})}{p^{(k+1)s}} + \cdots + \frac{S_k(p^{2k})}{p^{2ks}} + \cdots\right) =$$

$$\prod_p \left(1 + \frac{p}{p^s} + \cdots + \frac{p}{p^{ks}} + \frac{p^2}{p^{(k+1)s}} + \cdots + \frac{p^2}{p^{2ks}} + \cdots\right) =$$

$$\prod_p \left(1 + \frac{\frac{1}{p^s}(1 - \frac{1}{p^{ks}})}{1 - \frac{1}{p^s}} (p + \frac{p^2}{p^{ks}} + \frac{p^3}{p^{2ks}} + \cdots\right) =$$

$$\prod_p \left(1 + \frac{1 - \frac{1}{p^{ks}}}{p^s - 1} \times \frac{p}{1 - \frac{p}{p^{ks}}}\right) =$$

$$Y(s) \prod_p \frac{(p^s - 1)(p^{ks} - p) + p(p^{ks} - 1)}{p^s(p^{ks} - p)} =$$

$$Y(s) Y(ks - 1) \prod_p (1 + p^{1-s} - p^{1-ks} - p^{-s})$$

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其中 $\zeta(s)$ 为 Riemann Zeta函数

因 $|S_k(n)| \le n$

$$\left|\sum_{n=1}^{\infty}\frac{S_k(n)}{n^{\sigma}}\right| \le \frac{1}{(\sigma-1-b)^{\alpha}}\quad \alpha>0$$

这里 σ是 s的实部且大于 $1+b$,利用 perron公式[2],我们有

$$\sum_{n\le x}\frac{S_k(n)}{n^{s_0}} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) + O\left(x^{1-\sigma_0}H(2x)\min\left(1,\frac{\log x}{T}\right)\right) + O\left(x^{-\sigma_0}H(N)\min\left(1,\frac{x}{\|x\|}\right)\right)$$

其中 N 是离 x 最近的整数,且 $\|x\| = |x-N|$,

令: $s_0=0, b=1+\frac{1}{\ln x}, T=x^{\frac12}$,

$$H(x)=x,\ B(\sigma)=\frac{1}{(\sigma-1-b)^{\alpha}}$$

则我们有

$$\sum_{n\le x} S_k(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)\zeta(ks-1)g(s)\frac{x^s}{s}ds + O(x^{\frac12+\varepsilon})$$

其中 $g(s)=\prod_p(1+p^{1-s}-p^{1-ks}-p^{-s})$

下面我们估计主项:

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta(s)\zeta(ks-1)g(s)\frac{x^s}{s}ds$$

为此,令 $a=\frac12+\frac{1}{\ln x}$,并且把积分限从 $s=b\pm iT$ 移到 $s=a\pm iT$ 上,这时,当 $k\ge 3$时,

$$F(s)=\zeta(s)\zeta(ks-1)g(s)\frac{x^s}{s}$$

在 $s=1$处有一级极点且

$$\operatorname*{Res}_{s=1}F(s)=\zeta(k-1)g(1)x,$$

而余项

$$\frac{1}{2\pi i}\left(\int_{b+iT}^{a+iT}+\int_{a+iT}^{a-iT}+\int_{a-iT}^{b-iT}\right)\zeta(s)\zeta(ks-1)g(s)\frac{x^s}{s}ds \ll x^{\frac12+\varepsilon}$$

于是我们有

$$\sum_{n\le x} S_k(n) = \zeta(k-1)g(1)x + O(x^{\frac12+\varepsilon}) \qquad (1)$$

当 $k\le 2$时,

$$F(s)=\zeta(s)\zeta(ks-1)g(s)\frac{x^s}{s}$$

在极点 $s=\frac2k$处有残数$\frac{k}{2}\zeta(\frac2k)g(\frac2k)x^{\frac2k}$

类似于(1),我们得

$$\sum_{n\le x} S_k(n) = \frac{k}{2}\zeta(\frac2k)g(\frac2k)x^{\frac2k} + O(x^{\frac2k+\varepsilon}) \qquad (2)$$

证毕.

引理 **2**　对任意实数 $x\ge 1, k\ge 3, l\ge 1$,我们有 $\sum_{n\le x} n^l S_k(n) = \frac{1}{l+1}\zeta(k-1)g(1)x^{1+l} + O(x^{\frac12+l+\varepsilon})$

证明:　利用阿贝尔等式[3],我们有

$$\sum_{n\le x} n^l S_k(n) = x\sum_{n\le x}S_k(n) - \int_1^x l t^{l-1}\sum_{n\le t}S_k(t)dt = \zeta(k-1)g(1)x^{1+l} + O(x^{\frac12+l+\varepsilon}) - \int_1^x \zeta(k-1)g(1)lt^l dt + O\left(\int_1^x lt^{l-\frac12+\varepsilon}\right) = \frac{1}{l+1}\zeta(k-1)g(1)x^{1+l} + O(x^{\frac12+l+\varepsilon}) \qquad (3)$$

证毕.

3　定理的证明

现在我们证明定理 1

$$\sum_{n\le x}S_k(a_k(n)) = \sum_{1^k\le j<2^k}S_k(1) + \sum_{2^k\le j<3^k}S_k(2)+\cdots+\sum_{N^k\le j\le x<(N+1)^k}S_k(N) = \sum_{i=1}^{N-1}\sum_{i^k\le j<(i+1)^k}S_k(i) + \sum_{N^k\le j\le x<(N+1)^k}S_k(N) = \sum_{i=1}^{N-1}((i+1)^k-i^k)S_k(i) + \sum_{N^k\le j\le x<(N+1)^k}S_k(N) = \sum_{i=1}^{N-1}\sum_{l=0}^{k-1}C_k^l i^l S_k(i) + O(x^{\frac1k}) = \sum_{l=0}^{k-1}C_k^l\sum_{i\le x^{\frac1k}-1} i^l S_k(i) + O(x^{\frac1k}) = \sum_{l=0}^{k-1}C_k^l\left[\frac{1}{l+1}\zeta(k-1)g(1)x^{\frac{1+l}{k}} + O(x^{\frac{1}{2k}+\frac lk+\varepsilon})\right] + O(x^{\frac1k}) = \frac1k\zeta(k-1)g(1)x + O(x^{1-\frac{1}{2k}+\varepsilon})$$

定理 2的证明与定理 1类似,从而完成了定理的证明.

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The smarandache ceil function of order k and k-th roots of positive interger

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Abstract The mean value properties of the smarandache ceil function of order k acting on the k-th roots sequences were studied by using the analytic methods. Two interesting asymptotic formula were given.

Key words smarandache ceil function; k-th roots sequences; asymptotic formula

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$$\left[\frac{N-2}{(1-t)^3}-\frac{N-3}{(1-t)^2}\right]^k$$

将(2)式代入上式可得

$$\left[\sum_{n_1+n_2+\cdots+n_k=n} a_N(n+1)t^n\right]^k=$$

$$\sum_{n=0}^{\infty}\sum_{i=0}^{k} C_{n+2k+i-1}^{2k+i-1} C_k^i (3-N)^{k-i}(N-2)^i \qquad (11)$$

而 $$\left[\sum_{n=0}^{\infty} a_N(n+1)t^n\right]^k=$$

$$\sum_{n=0}^{\infty}\left[\sum_{n_1+n_2+\cdots+n_k=n} a_N(n_1+1)\, a_N(n_2+1)\cdots a_N(n_k+1)\right]t^n \qquad (12)$$

比较(11)及(12)式中的 t^n 的系数便可得到定理 4的结论.

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The combinatorial identities of some sequences

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Abstract Some properties of the sequences were studied by using the elementary methods. And some combinatorial identities were obtained.

Key words sequences; combinatorial identity; elementary methods

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The identical relations about n-order Bernoulli Numbers

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Abstract Using the definitions of n-order Bernoulli Numbers and the Stirling Numbers of the first kind and second kind, the relations betweer them were studied, some identical relations betweer Bernoulli Numbers and Stirling Numbers were obtained.

Key words identical relation; n-order Bernoulli Numbers; Stirling Numbers